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SO(3, 1)-Invariant Approach to Dipole Radiation

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Abstract The aim of the present work is to revisit the theory of the dipole radiation, within an $SO(3,1)$-gauge invariant formulation, by solving the Maxwell equations. Thus, we obtain the two interconnected components, A_B, $B = 1, 2$, of the vector potential $\mathbf{A}$, in terms of Hankel and Legendre polynomials. Finally, for the *pure* dipole-like radiation, the observables, regarded as phasors, the Umov–Poynting vector components and the well-known Larmor formula for the effective radiated power are explicitly derived.

Keywords Maxwell equations · Dipole radiation

1 Introduction

Even though the problem of a particle moving in given external fields is classical, it has always attracted attention. As it is well-known, the radiation of a heavy charged accelerated particle is generally treated, in classical electrodynamics, by computing the Lienard–Wiechert retarded potentials, [9]. However, due to the recent development in a wide range of theoretical fields, as for example in the quark-gluon plasma, the propagation of particles in external fields has become a real target of investigations. In General Relativity, the electromagnetic radiation of an electric dipole in the center of a spherical envelope has been initiated, almost 30 years ago, [11], and has been an interesting subject ever since. Later, Torres del Castillo and Alcazar-Olan have studied the dipole radiation in the context of scattering by a Reissner-Nordstrom black hole, [5], B. Blok was interested by the radiation reaction and influence on the spacetime evolution of small relativistic dipole moving in a constant external electromagnetic field, [1], and a simple experiment for testing the concept of gravitational radiation source emitted from an oscillating gravitational dipole has been proposed, [2–4]. Moreover, in the context of string theories and D-branes, it has been shown that a string attached to the 3-brane manifests itself as an electric charge and waves

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on the string produce electromagnetic dipole radiation in the asymptotic outer region, with the power equal to the one given by the classical Thomson formula, [12].

With the advent of the impressive results coming from the CMBR, in FRW cosmologies, the origin of the cosmological dipole has become a challenging subject. It has been suggested that, for an open cosmology, the observed CMBR dipole might be coming from ultra-large scale isocurvature perturbations. In this context, it was shown that in order to get a total dipole 100 times bigger than a quadrupole, the curvature scale has to be 100 times larger than the Hubble radius, so it is extremely close to a flat Universe, [10]. Moreover, in the light of the recent experimental data of GRAAL facility located in Grenoble, it has been initiated a study of the light speed anisotropy with respect to the direction of the CMB dipole, [7]. The general problematic has deep implications since, besides being a cosmological messenger, the CMBR defines the frame where the dipole and the quadrupole anisotropies vanish as the *absolute* inertial frame of rest.

In this general context, we believe that the analysis of the dipole radiation, within an $SO(3,1)$-gauge invariant formulation, by solving the Maxwell equations, in Minkowski spacetime, is far from conventional. Apart from the internal beauty, there is still a lot of interest in the subject, since it can be seen as a preparation for much difficult cases, as for example the vertical dipole radiation in Milne Universe, [8].

2 Dipole Radiation in Spherical Coordinates

Let us consider the flat metric

$$ds^2_{(0)} = (dr)^2 + r^2 d\Omega^2 - (d\tau)^2, \qquad d\Omega^2 = (d\theta)^2 + \sin^2\theta (d\varphi)^2, \tag{1}$$

and the Maxwell equations

$$F^{ab}_{\ ;b} = 0, \quad \text{where } F_{ab} = A_{b;a} - A_{a;b}, \tag{2}$$

subjected to the $U(1)$-gauge fixing condition

$$A^a_{;a} = 0, \quad \text{with } A^4 = 0 = A^3 \text{ and } \frac{\partial A^A}{\partial\varphi} = 0, \tag{3}$$

where the index $A = 1, 2$. The semi-colons stand for the Levi–Civita covariant derivatives with respect to the metric (1). The corresponding coefficients of the Levi–Civita connection, Γ, for the metric (1),

$$ds_{(0)} = \eta_{ab}\Omega^a\Omega^b, \tag{4}$$

where $(\eta_{ab}) = \text{diag}[1, 1, 1, -1]$ and

$$\Omega^1 = dr, \qquad \Omega^2 = r d\theta, \qquad \Omega^3 = r\sin\theta d\varphi, \qquad \Omega^4 = d\tau, \tag{5}$$

come to the first Cartan equation (without torsion)

$$d\Omega^a = \Gamma^a_{\cdot[bc]}\Omega^b \wedge \Omega^c, \quad \text{where } 1 \le b < c \le 4 \text{ and } \Gamma^a_{\cdot[bc]} = \Gamma^a_{\cdot bc} - \Gamma^a_{\cdot cb}, \tag{6}$$

with the metric condition $\Gamma_{(ab)c} = 0$, where $\Gamma_{(ab)c} = \Gamma_{abc} + \Gamma_{bac}$. Therefore, it yields the essential coefficients

$$\Gamma_{212} = \frac{1}{r} = \Gamma_{313} \quad \text{and} \quad \Gamma_{323} = \frac{\cot\theta}{r}, \tag{7}$$

with $\Gamma_{122}=\Gamma_{133}=-\Gamma_{212}$ and $\Gamma_{233}=-\Gamma_{323}$, where $\Gamma_{abc}=\eta_{ad}\Gamma^{d}_{\cdot bc}$. Accordingly, under the gauge-fixing condition (3), the only non-vanishing components of the Maxwell tensor **F** read

$$F_{12}=A_{2|1}-A_{1|2}+\Gamma_{212}A_2, \qquad F_{14}=-A_{1|4}, \qquad F_{24}=-A_{2|4} \tag{8}$$

and subsequently, the essential Maxwell equations (2),

$$F^{ab}_{;b}=F^{ab}_{|b}+F^{cb}\Gamma^{a}_{\cdot cb}+F^{ac}\Gamma^{b}_{\cdot cb}=0, \tag{9}$$

do actually means

$$F_{12|2}-F_{14|4}+\Gamma_{323}F_{12}=0; \tag{10a}$$

$$F_{12|1}+F_{24|4}+\Gamma_{313}F_{12}=0; \tag{10b}$$

$$F_{14|1}+F_{24|2}+(\Gamma_{212}+\Gamma_{313})F_{14}+\Gamma_{323}F_{24}=0. \tag{10c}$$

Meanwhile, the gauge-fixing condition (3) comes to the primary form

$$A_{1|1}+(\Gamma_{212}+\Gamma_{313})A_1+A_{2|2}+\Gamma_{323}A_2=0. \tag{11}$$

Since the $U(1)$-gauge potential **A** has only two non-vanishing components which are subjected to the four differential equations (10) and (11), it yields that two of the later should be redundant. Indeed, for the first redundancy, let us write down in full the Maxwell equation (10c) and the Lorentz condition (11), i.e.

$$\frac{\partial}{\partial r}\left(\frac{\partial A_1}{\partial t}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial A_2}{\partial t}\right)+\frac{2}{r}\frac{\partial A_1}{\partial t}+\frac{\cot\theta}{r}\frac{\partial A_2}{\partial t}=0; \tag{12a}$$

$$\frac{\partial A_1}{\partial r}+\frac{2}{r}A_1+\frac{1}{r}\frac{\partial A_2}{\partial\theta}+\frac{\cot\theta}{r}A_2=0. \tag{12b}$$

As it can be noticed, the first one of them is nothing else then

$$\frac{\partial}{\partial t}\left[\frac{\partial A_1}{\partial r}+\frac{2}{r}A_1+\frac{1}{r}\frac{\partial A_2}{\partial\theta}+\frac{\cot\theta}{r}A_2\right]=0 \tag{13}$$

and because of the second one, it trivially yields zero. The case for the second redundancy, involving (10b), is a little more complicated. Based on the Lorentz condition (12b), one firstly has to derive the propagation equation of the component A_1. After a bit of calculation, this can be done, writing the Maxwell equation (10a) under the form

$$\frac{1}{r^2}\left[\frac{\partial^2 A_1}{\partial\theta^2}+\cot\theta\frac{\partial A_1}{\partial\theta}\right]-\frac{\partial^2 A_1}{\partial t^2}=\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\partial A_2}{\partial\theta}+\cot\theta A_2\right]+\frac{1}{r^2}\left(\frac{\partial A_2}{\partial\theta}+\cot\theta A_2\right). \tag{14}$$

Inserting here the Lorentz condition (12b), one finally gets the very important equation,

$$\frac{\partial^2 A_1}{\partial r^2}+\frac{4}{r}\frac{\partial A_1}{\partial r}+\frac{2}{r^2}A_1+\frac{1}{r^2}\left(\frac{\partial^2 A_1}{\partial\theta^2}+\cot\theta\frac{\partial A_1}{\partial\theta}\right)-\frac{\partial^2 A_1}{\partial t^2}=0, \tag{15}$$

controlling the radial A_1-component propagation for the case of vertical dipole radiation in flat spacetime. Now, one can switch to the second Maxwell equation (10b),

$$\frac{\partial}{\partial r}\left[\frac{\partial A_2}{\partial r}-\frac{1}{r}\frac{\partial A_1}{\partial\theta}+\frac{1}{r}A_2\right]-\frac{\partial^2 A_2}{\partial t^2}+\frac{1}{r}\left[\frac{\partial A_2}{\partial r}-\frac{1}{r}\frac{\partial A_1}{\partial\theta}+\frac{1}{r}A_2\right]=0, \tag{16}$$

which concretely becomes

$$\frac{\partial^2 A_2}{\partial r^2} + \frac{2}{r}\frac{\partial A_2}{\partial r} - \frac{\partial^2 A_2}{\partial t^2} = \frac{1}{r}\frac{\partial^2 A_1}{\partial r \partial\theta}. \tag{17}$$

Because the Lorentz condition (12b) can also be written as

$$\frac{\partial}{\partial\theta}[\sin\theta A_2] = -r\sin\theta\left[\frac{\partial A_1}{\partial r} + \frac{2}{r}A_1\right], \tag{18}$$

one can multiply (17) by $\sin\theta$ and subsequently take the ∂_θ-derivative. Thus, it yields the equation

$$r\frac{\partial^3 A_1}{\partial r^3} + 6\frac{\partial^2 A_1}{\partial r^2} + \frac{6}{r}\frac{\partial A_1}{\partial r} - r\frac{\partial}{\partial r}\left(\frac{\partial^2 A_1}{\partial t^2}\right) - 2\frac{\partial^2 A_1}{\partial t^2}$$
$$= -\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\partial^2 A_1}{\partial\theta^2} + \cot\theta\frac{\partial A_1}{\partial\theta}\right], \tag{19}$$

which, because of (15), reads

$$6\frac{\partial^2 A_1}{\partial r^2} + \frac{6}{r}\frac{\partial A_1}{\partial r} = 2\frac{\partial^2 A_1}{\partial r^2} + 4\frac{\partial^2 A_1}{\partial r^2} + \frac{4}{r}\frac{\partial A_1}{\partial r} + \frac{2}{r}\frac{\partial A_1}{\partial r} \tag{20}$$

and comes indeed to the trivial identity $0 \equiv 0$ which proves the second redundancy.

Therefor, in brief, the following conclusion can be drawn. The axially symmetric radiation of a vertical dipole in spherically parametrized Minkowski spacetime is directly described by the two interconnected components A_B, $B = 1, 2$, of the vector potential $\mathbf{A}$, solutions to the expressions (15) and (18), i.e.

$$A_2 = -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}\int [r^2 A_1 \sin\theta] d\theta \tag{21}$$

and by the corresponding observables

$$B_3 \triangleq F_{12} = \frac{\partial A_2}{\partial r} + \frac{1}{r}A_2 - \frac{1}{r}\frac{\partial A_1}{\partial\theta},$$
$$E_1 \triangleq F_{14} = -\frac{\partial A_1}{\partial t}, \tag{22}$$
$$E_2 \triangleq F_{24} = -\frac{\partial A_2}{\partial t},$$

which stand for the essential components of the magnetic induction and of the electric field intensity, respectively.

As all of the involved equations (15), (21) and (22), are linear with respect to the $A_B(r, \theta, t)$-components, the goal would be to work out the so-called *model solutions*, $A_{\omega\ell B}(r, \theta, t)$, since any other solution of the Maxwell system (15, 21) will be given by a complex linear combination of these modes. Therefore, we start with the (linear) differential equation (15), wherein we look for the field solutions which can be expressed as

$$A_1(r, \theta, t) = F(r)\Theta(\theta)T(t). \tag{23}$$

Accordingly, the equation becomes

$$\frac{1}{F}\left[\frac{d^2F}{dr^2}+\frac{4}{r}\frac{dF}{dr}+\frac{2}{r^2}\right]+\frac{1}{r^2\Theta}\left[\frac{d^2\Theta}{d\theta^2}+\cot\theta\frac{d\Theta}{d\theta}\right]-\frac{1}{T}\frac{d^2T}{dt^2}=0, \tag{24}$$

so that is getting clear that the regular harmonic part in θ and t comes out as the solutions

$$\Theta(\theta)=Y_\ell^0(\theta)=\sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta), \qquad T(t)=\{e^{i\omega t},e^{-i\omega t}\} \tag{25}$$

to the eigenvalue problems

$$\frac{d^2\Theta}{d\theta^2}+\cot\theta\frac{d\Theta}{d\theta}+\ell(\ell+1)\Theta=0 \quad\text{and}\quad \frac{d^2T}{dt^2}+\omega^2T=0, \tag{26}$$

where $\ell\in\mathbf{N}$, $\omega\in\mathbf{R}_+$ and P_ℓ stands for the Legendre polynomial of degree ℓ. Thence, the radial amplitude function equation becomes

$$\frac{d^2F}{d\rho^2}+\frac{4}{\rho}\frac{dF}{d\rho}+\left[1-\frac{(\ell-1)(\ell+2)}{\rho^2}\right]F=0, \tag{27}$$

where we have already defined the physically dimensionless radial coordinate $\rho=\omega r$. Because of the singularity in $\rho=0$, one has to employ the function substitution

$$F(\rho)=\rho^\alpha H(\rho), \quad \alpha\in\mathbf{R}, \tag{28}$$

so that, with respect to the new function $H(\rho)$, the equation turns, for $\alpha=-3/2$, into the Bessel equation

$$\frac{d^2H}{d\rho^2}+\frac{1}{\rho}\frac{dH}{d\rho}+\left[1-\frac{(\ell+1/2)^2}{\rho^2}\right]H=0, \tag{29}$$

with the pure *out*-like solution

$$H(\rho)=H^{(1)}_{\ell+\frac{1}{2}}(\rho), \tag{30}$$

where the function on the right stands for the first Hankel function of order $\ell+\frac{1}{2}$. Thus, putting everything together and including a *constant* amplitude factor $\mathcal{N}$, the positive frequency out-going mode-solution for the component A_1 does concretely read

$$A^{(em)}_{w\ell 1}=\sqrt{\frac{2\ell+1}{4\pi}}\frac{\mathcal{N}}{(\omega r)^{3/2}}H^{(1)}_{\ell+\frac{1}{2}}(\omega r)P_\ell(\cos\theta)e^{-i\omega t} \tag{31}$$

and subsequently, one gets from (21), the corresponding modal solution

$$A^{(em)}_{w\ell 2}=-\sqrt{\frac{2\ell+1}{4\pi}}\frac{\mathcal{N}}{r\sin\theta}\frac{d}{dr}\left[\frac{r^2}{(\omega r)^{3/2}}H^{(1)}_{\ell+\frac{1}{2}}(\omega r)\right]\left(\int_{\cos\theta}^1 P_\ell(\xi)d\xi\right)e^{-i\omega t}. \tag{32}$$

Since, according to [6],

$$\int_{\cos\theta}^1 P_\ell(\cos\beta)d(\cos\beta)=\sin\theta\cdot P_\ell^{-1}(\cos\theta)=\frac{P_{\ell-1}(\cos\theta)-P_{\ell+1}(\cos\theta)}{2\ell+1}, \tag{33}$$

the positive frequency *out*-mode solution for the component A_2 becomes

$$A^{(em)}_{w\ell 2} = -\sqrt{\frac{2\ell+1}{4\pi}}\frac{\mathcal{N}}{r}\frac{d}{dr}\left[\frac{r^2}{(\omega r)^{3/2}}H^{(1)}_{\ell+\frac{1}{2}}(\omega r)\right] \times \frac{P_{\ell-1}(\cos\theta)-P_{\ell+1}(\cos\theta)}{(2\ell+1)\sin\theta}e^{-i\omega t}. \tag{34}$$

Now, the real fact is that, because of the Hankel functions, their derivatives and the Legendre polynomials of various degrees, things are getting too general with respect to the structure of the radiating (electromagnetic) field, in terms of arbitrary natural values of the quantum number ℓ, and therefore, we are going to restrict the analysis to the case $\ell = 1$, which goes by the name of *pure* dipole-like radiation. Dropping the unnecessary specifications, in order to simplify the writing, we come to the following situation

$$\begin{aligned} &H^{(1)}_{\frac{3}{2}}(\rho) = -\sqrt{\frac{2}{\pi\rho}}\left(1+\frac{i}{\rho}\right)e^{i\rho}, \\ &P_0(\cos\theta) = 1, \quad P_1(\cos\theta) = \cos\theta, \quad P_2(\cos\theta) = \frac{1}{2}[3\cos^2\theta - 1]. \end{aligned} \tag{35}$$

According to (31) and (34), for $\ell = 1$, we use the concrete expressions (35) and taking the derivative involved with the A_2-component, one gets in clear the essential components of the $U(1)$-gauge connection $A = A_a\Omega^a$, $a = \overline{1,4}$, for $\ell = 1$,

$$\begin{aligned} A_1 &= -\frac{\sqrt{3/2}}{\pi}\frac{\mathcal{N}}{\rho^2}\left(1+\frac{i}{\rho}\right)e^{i\rho}\cos\theta e^{-i\omega t}, \\ A_2 &= i\frac{\sqrt{3/2}}{2\pi}\frac{\mathcal{N}}{\rho}\left[1+\frac{i}{\rho}-\frac{1}{\rho_2}\right]e^{i\rho}\sin\theta\cdot e^{-i\omega t}. \end{aligned} \tag{36}$$

These lead, by (8), to the *observables*—regarded as phasors—

$$\begin{aligned} B_3 &= -\frac{\sqrt{3/2}}{2\pi}\frac{\omega\mathcal{N}}{\rho}\left(1+\frac{i}{\rho}\right)e^{i\rho}\sin\theta e^{-i\omega t}, \\ E_1 &= -i\frac{\sqrt{3/2}}{\pi}\frac{\omega\mathcal{N}}{\rho^2}\left(1+\frac{i}{\rho}\right)e^{i\rho}\cos\theta e^{-i\omega t}, \\ E_2 &= -\frac{\sqrt{3/2}}{2\pi}\frac{\omega\mathcal{N}}{\rho}\left(1+\frac{i}{\rho}-\frac{1}{\rho_2}\right)e^{i\rho}\sin\theta e^{-i\omega t}. \end{aligned} \tag{37}$$

3 The Umov–Poynting Vector and Radiated Power

Now, we must call in the International Units system, get the physical meaning of the amplitude factor $\mathcal{N}$, in terms of some observables straightly related to the source of the electromagnetic radiation, and to explicitly work out the expression of the radiated power in order to be convinced that we are dealing with an electric dipole. These can be done noticing at first—from (36)—that the dimension of $\mathcal{N}$ must be Tesla $\cdot$ meter $=$ kg $\cdot$ m $\cdot$ s$^{-2}\cdot$ A^{-1}. Thence, since $\omega = ck$, E (V/m) $= c$ (m/s) $\cdot B$ (Tesla) and $\rho = kr$ stands for the actual dimensionless radial coordinate, it yields the actual radiating field configuration:

$$
\begin{aligned}
E_1 &= -i \frac{c\sqrt{\frac{3}{2}}\mathcal{N}}{\pi k r^2}\left[1 + \frac{i}{kr}\right]\cos\theta e^{i(kr-\omega t)}, \\
E_2 &= -\frac{c\sqrt{\frac{3}{2}}\mathcal{N}}{2\pi r}\left[1 + \frac{i}{kr} - \frac{1}{k^2 r^2}\right]\sin\theta e^{i(kr-\omega t)}, \\
B_3 &= -\frac{\sqrt{\frac{3}{2}}\mathcal{N}}{2\pi r}\left[1 + \frac{i}{kr}\right]\sin\theta e^{i(kr-\omega t)}.
\end{aligned}
\tag{38}
$$

Now, the best would be to surround the origin $r = 0$ with a sphere of radius a and to invoke the Gauss Theorem,

$$
\varepsilon_0 \oint_{S^2_+[a]\cup D[a]} \vec{E}\cdot d\vec{\sigma} = q_+(t;a),
$$

on the northern hemisphere closed below by the equatorial disk, $\theta = \pi/2$, of radius a, in order to find out the *effective* charge setting therein. If we did that, with

$$
\begin{aligned}
d\sigma_1 &= [\Omega^2 \wedge \Omega^3]_{r=a} = a^2 \sin\theta d\theta d\varphi, \\
d\sigma_2 &= [\Omega^3 \wedge \Omega^1]_{\theta=\pi/2} = r dr d\varphi,
\end{aligned}
\tag{39}
$$

it yields

$$
\begin{aligned}
q_+(t;a) = \varepsilon_0 \Bigg\{ &\int_0^{\pi/2}\int_0^{2\pi} E_1(r = a, \theta, t)\, a^2 \sin\theta d\theta d\varphi \\
&+ \int_0^a \int_0^{2\pi} E_2\left(r, \theta = \frac{\pi}{2}, t\right) r dr d\varphi \Bigg\}
\end{aligned}
\tag{40}
$$

i.e.

$$
q_+(t;a) = -\varepsilon_0 c\sqrt{\frac{3}{2}}\mathcal{N}\left\{\frac{i}{k}\left[1 + \frac{i}{ka}\right]e^{ika} + \int_0^a \left[1 + \frac{i}{kr} - \frac{1}{k^2 r^2}\right]e^{ikr} dr\right\} e^{-i\omega t}. \tag{41}
$$

As it can be noticed, defining the function

$$
F(r) = \frac{i}{k}\left[1 + \frac{i}{kr}\right]e^{ikr}, \tag{42}
$$

its derivative reads

$$
\frac{dF}{dr} = -\left[1 + \frac{i}{kr} - \frac{1}{k^2 r^2}\right]e^{ikr}, \tag{43}
$$

so that the integral involved with the second term in the r.h.s. of (41) does actually become

$$
\int_0^a \left[1 + \frac{i}{kr} - \frac{1}{k^2 r^2}\right]e^{ikr} dr = -F(a) + \lim_{h\to 0_+} F(h). \tag{44}
$$

Plugging it back, into (41), it yields the a-independent expression

$$q_+(t) = -\varepsilon_0 c \lim_{h\to 0_+} \left[\frac{i\sqrt{\frac{3}{2}}\mathcal{N}}{k} \left(1 + \frac{i}{kh}\right) e^{ikh} \right] e^{-i\omega t}$$
$$= \varepsilon_0 c \frac{\sqrt{\frac{3}{2}}\mathcal{N}}{k^2 h} e^{-i\omega t} \quad \text{with } h \to 0_+, \tag{45}$$

since

$$\left(1 + \frac{i}{kh}\right) e^{ikh}\Big|_{h\to 0_+} = \left(1 + \frac{i}{kh}\right)(1 + ikh)\Big|_{h\to 0_+} = 1 + \frac{i}{kh} - 1 = \frac{i}{kh}, \tag{46}$$

which means indeed that in the upper region of $\mathbf{R}^3$, along Oz, there is an oscillating point-like charge at the very short distance h from the origin. Thence, denoting the maximal value of this charge by q_0, so that

$$q_+(t) = q_0 e^{-i\omega t}, \quad \text{with } q_0 \in \mathbf{R}_+, \tag{47}$$

it yields, from the expression (45), the important relation

$$\sqrt{\frac{3}{2}}\mathcal{N} = \frac{k^2}{c\varepsilon_0} q_0 h, \tag{48}$$

which states the physical meaning of the amplitude factor $\mathcal{N}$. Thus, inserting (48) into the observables (38), one comes to the concrete structure of the radiating electro-magnetic field

$$E_1 = -i \frac{k q_0 h}{\pi \varepsilon_0 r^2} \left[1 + \frac{i}{kr}\right] \cos\theta e^{i(kr-\omega t)}, \tag{49a}$$

$$E_2 = -\frac{k^2 q_0 h}{2\pi \varepsilon_0 r} \left[1 + \frac{i}{kr} - \frac{1}{k^2 r^2}\right] \sin\theta e^{i(kr-\omega t)}, \tag{49b}$$

$$B_3 = -\frac{k^2 q_0 h}{2\pi \varepsilon_0 c r} \left[1 + \frac{i}{kr}\right] \sin\theta e^{i(kr-\omega t)}. \tag{49c}$$

In order to be sure that we are dealing with an electric dipole, the static limit $\omega = ck \to 0_+$, where (obviously) B_3 is given by (49c) vanishes, can be taken, leaving us with the equations—from the expressions (49a, b)—

$$\frac{\partial V_s}{\partial r} = -E_1|_{k=0} = -\frac{q_0 h}{\pi \varepsilon_0} \cdot \frac{\cos\theta}{r^3},$$
$$\frac{\partial V_s}{\partial \theta} = -r E_2|_{k=0} = -\frac{q_0 h}{2\pi \varepsilon_0} \cdot \frac{\sin\theta}{r^2},$$

whose common exact solution,

$$V_s(\vec{r}, t) = \frac{\vec{p}\cdot\vec{r}}{4\pi \varepsilon_0 r^3}, \tag{50}$$

does clearly describe the electrostatic potential V_s of a vertical electric dipole, of momentum $\vec{p} = q_0 d\vec{u}_z$, where $d = 2h$ is nothing else than the distance interval along Oz between the two opposite charges $\pm q_0$.

Finally, in order to compute the effective radiated power, one firstly needs the explicit form of the time-averaged Umov–Poynting vector, which is defined for harmonic regimes, within the complex electromagnetic field formalism, by the expression

$$\vec{\Pi} = \frac{1}{4\mu_0}\{\vec{E} \times \bar{\vec{B}} + \bar{\vec{E}} \times \vec{B}\}. \tag{51}$$

Since, in our case,

$$\begin{aligned} &\vec{E} = E_A \vec{e}_A \quad \text{(sum over A} = 1, 2), \\ &\vec{B} = B_3 \vec{e}_3 \quad \text{and} \\ &\vec{e}_\mu \times \vec{e}_\nu = \varepsilon_{\mu\nu\lambda} \vec{e}_\lambda, \end{aligned} \tag{52}$$

where the greek indices run from 1 to 3, ε being the $3D$-completely antisymmetric Levi–Civita tensor with $\varepsilon_{123} = +1$, it yields, from (51), the *definition* relation

$$\vec{\Pi} = \frac{1}{4\mu_0}\varepsilon_{AB}(E_B \bar{B}_3 + \bar{E}_B B_3)\vec{e}_A, \tag{53}$$

where $\varepsilon_{AB} = \varepsilon_{AB3}$ stand for the essential components of the $2D$-Levi–Civita tensor, with $\varepsilon_{12} = +1$. Therefore, envisaging the complex electromagnetic field *observables* (49), it yields

$$\Pi_2 = -\frac{1}{4\mu_0}(E_1 \bar{B}_3 + \bar{E}_1 B_3) \equiv 0, \tag{54}$$

so that it only survives the radial component

$$\Pi_1 = \frac{1}{4\mu_0}(E_2 \bar{B}_3 + \bar{E}_2 B_3), \tag{55}$$

which is actually given by the full expression

$$\Pi_1 = \frac{\omega^4 p_0^2}{32\pi^2 c^3 \varepsilon_0} \cdot \frac{\sin^2\theta}{r^2}, \tag{56}$$

where $p_0 = q_0 d$ and $d = 2h$. Thence, the effective radiated power, generally given by the expression

$$P = \oint_{S^2[\infty]} \vec{\Pi} \cdot d\vec{\sigma}, \quad \text{where } d\vec{\sigma} = \frac{1}{2}\varepsilon^{\alpha}_{\cdot\beta\gamma}\Omega^\beta \wedge \Omega^\gamma \vec{e}_\alpha, \tag{57}$$

does only become

$$P = \int_{(4\pi)} r^2 \Pi_1 \sin\theta d\theta d\varphi = \frac{\omega^4 (q_0 d)^2}{16\pi\varepsilon_0 c^3}\int_0^\pi \sin^3\theta d\theta = \frac{\omega^4 (q_0 d)^2}{12\pi\varepsilon_0 c^3}, \tag{58}$$

being precisely the more generally written Larmor formula

$$P = \frac{\langle |\ddot{\vec{p}}|^2 \rangle}{6\pi\varepsilon_0 c^3}, \tag{59}$$

for the case of the dipole radiation.

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